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# The Goldstone Boson Equivalence Theorem

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The biggest topic in particle physics today is the Higgs boson. We are excited to have discovered this particle, 35 years after its prediction as a key element of the theory of weak interactions. Now, one of the main purposes of the LHC experiment—and an important driver for experiments at future high-energy colliders—is the precision measurement of the properties of the Higgs boson. In particular, we want to know whether this particle has *exactly* the properties predicted by the Standard Model, or whether there is room for a more general "Higgs sector" containing many new particles. This question is closely linked to the direct search for new particles and forces at high energy. This linked set of questions — What is the origin of the spontaneous breaking of the weak interaction symmetry? Are there new particles and forces at TeV energies? —is one of the most important open questions in science. For me, it is *the* question for which I would like to know the answer.

One of the subtle aspects of the physics of the Higgs boson is that some of its properties are free and subject to experimental test while other properties are fixed by the requirement that the Higgs field should give mass to the particles of the Standard Model. In particular, the properties of the Higgs field are strongly constrained by the generation of masses for the $W$ and $Z$ bosons. The Higgs mechanism is the *only* way to give mass to vector bosons. "Only" here is a strong qualifier, but there are strong requirements that come from Lorentz invariance and unitarity. These requirements are expressed in the Ward Identities of the electroweak theory, but that is not an easy way to extract and make use of their information. A much more intuitive and physical way is to think about their implications for the emission of massive vector bosons at high energy. This is the subject of the "Goldstone Boson Equivalence Theorem".

The Goldstone Boson Equivalence Theorem (GBET) was first enunciated by Cornwall, Levin, and Tiktopoulos [1] and Vayonakis [2]. A clean and general proof of the theorem has been given by Chanowitz and Gaillard [3], in a beautiful paper that also discusses applications to high-energy scattering processes. In these lectures, I will not prove the theorem, but I will explain it and give some illustrative examples.

To begin, I would like to recall some statements that you learned when you were introduced to the Standard Model of weak interactions. This model is based on a picture of the $W$ and $Z$ fields as gauge fields obeying, as their equation of motion, a generalization of Maxwell's equations. In principle, we could have started with more general vector fields $A_\mu(x)$, with four degrees of freedom per space-time point. We could quantize these fields just as we quantize a scalar field, by introducing creation and annihilation operators $a_p^{\dagger\mu}$ and $a_p^\mu$. These would need to obey a Lorentz-invariant commutation relation

$$[a_p^\mu, a_k^{\nu\dagger}] = -g^{\mu\nu}(2\pi)^3\delta(p-k)$$

with

$$-g^{\mu\nu} = \left(\begin{array}{c|ccc} -1 & & & \\ \hline & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right)$$

The minus sign in the first equation compensates the minus sign in $g^{\mu\nu}$. This makes good sense for $\mu, \nu = 1, 2, 3$, but for $\mu, \nu = 0$ it is a disaster. The commutation relation implies that

$$\langle 0| a_p^0 a_p^{0\dagger} |0\rangle = \| a_p^{0\dagger}|0\rangle \|^2 < 0$$

so that the theory has negative norms and, in quantum mechanics, negative probabilities. We need to eliminate the troublesome states, and we need to do this in a Lorentz-invariant way.

Maxwell's equations provide a nice solution to this problem. The solutions to Maxwell's equations are plane waves that move at the speed of light. Each wave has a definite momentum $p$ and, especially, a definite direction of propagation. Of the four possible space-time directions, only two give solutions to Maxwell's equations. For $\vec{p} \parallel \hat{3}$,

$$\varepsilon^\mu = (0\ 1\ 0\ 0)$$

or $$\varepsilon^\mu = (0\ 0\ 1\ 0)$$

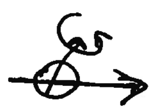
In this lecture, I will often prefer to work with the linear combinations

$$\varepsilon_L^\mu = \frac{1}{\sqrt{2}} \left( 0 \; 1 \; -i \; 0 \right)$$

$$\varepsilon_R^\mu = \frac{1}{\sqrt{2}} \left( 0 \; 1 \; i \; 0 \right)$$

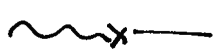
corresponding to left- and right-handed circularly polarized light or photon helicity $-1$ and $+1$, respectively.

This solves the problem of eliminating the troublesome states for *massless* spin-1 particles but not for *massive* spin-1 particles. For a massive spin-1 particle, we can always boost to its rest frame.

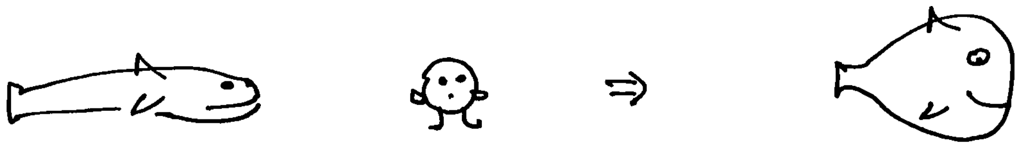


In the rest frame, we can rotate the polarization vector arbitrarily. Thus, we need 3 orthogonal polarization vectors to describe all of the particle states. It is equivalent to say that, in quantum mechanics, an elementary spin 1 particle has 3 quantum states, corresponding to $m = -1, 0, 1$. Maxwell's equations give us only two of these states; where is the third?

The Higgs mechanism solves this problem in the following way: We arrange that the symmetry breaking associated with the gauge boson is spontaneously broken. A spontaneously broken continuous symmetry gives rise to an associated Goldstone boson. This boson couplings to the gauge field as

 $\sim \; g f p^\mu$

When we now solve for the propagating states, the Goldstone boson has combined with the gauge boson and given it an extra degree of freedom. We say that the gauge boson has "eaten" the Goldstone boson and become massive. Kurt Gottfried illustrated this in the following way:

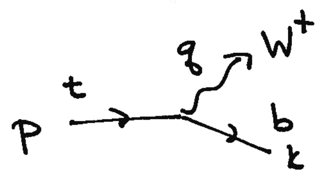


At this point, the Goldstone boson has disappeared from the physical spectrum of the theory. However, we can find a trace of its presence in amplitudes for the emission of very high energy vector bosons. In the high-energy limit, the mass of the vector boson should become irrelevant. But, the vector boson still has three degrees of freedom for each value of the momentum. It is reasonable to expect that the transverse polarization states, those with the polarization vectors $\epsilon_L^\mu$ and $\epsilon_R^\mu$ above, have the properties of the original massless vector bosons. The third state, which we describe as having a *longitudinal* polarization vector $\epsilon_0^\mu$, must have the properties of the Goldstone boson. This is the content of the Goldstone Boson Equivalence Theorem. For example, for a $W^+$ boson,

$$\mathcal{M}(X \to Y + W^+(p)) \xrightarrow[p^0/m_W \to \infty]{} \mathcal{M}(X \to Y + \pi^+(p))$$

where the corrections are of order $m_W/E_W$.

A simple example of the GBET comes from the theory of the decay of the top quark. The top quark is the heaviest quark and, as such, is allowed to decay directly to an on-shell $W$ boson and $b$ quark.




It is tempting to compute the rate of the top quark decay in the following way: We write the $W$ boson emission amplitude as

$$\mathcal{M} = i\frac{g}{\sqrt{2}}\, \bar{u}_b(k)\, \gamma^\mu \left(\frac{1-\gamma^5}{2}\right) u_t(p)\, \epsilon^*_\mu(q)$$

then square and sum over polarizations in the usual way, using

$$\sum_i \epsilon_i^\mu(q)\,\epsilon_i^{\nu *}(q) = -g^{\mu\nu}$$

to do the sum over vector boson polarizations. This gives

$$\frac{1}{2}\sum_{spin}|\mathcal{M}|^2 = \frac{g^2}{2}\frac{1}{2}\,\mathrm{tr}\left[\not k\,\gamma^\mu(\not p + m_t)\gamma^\nu\left(\frac{1-\gamma^5}{2}\right)\right](-g_{\mu\nu})$$

$$= -\frac{g^2}{4}\cdot 4\cdot(-2)\cdot\frac{1}{2}\cdot p\cdot k = g^2 m_t E_b = \frac{g^2}{2}(m_t^2 - m_W^2)$$

From this expression, we find

$$\Gamma_t = \frac{1}{2m_t}\,\frac{1}{8\pi}\left(1 - \frac{m_W^2}{m_t^2}\right)\,\frac{g^2}{2}(m_t^2 - m_W^2)$$

and, finally,

$$\Gamma_t = \frac{\alpha_w}{8}\,m_t\,\left(1 - \frac{m_W^2}{m_t^2}\right)^2 \qquad \alpha_w = \frac{g^2}{4\pi}$$

This answer looks good and meets out naive expectation that the decay rate should be of the order of

$$\Gamma_t \sim \alpha_w m_t$$

However, it is completely wrong.

The problem is that we have not done the sum over $W$ polarizations correctly. In the rest frame, the polarization vectors are the three spatial vectors. Then, in the rest frame,

$$\sum_{i=1..3} \varepsilon_i^{\mu} \varepsilon_i^{\nu *} = \left(\begin{array}{c|ccc} 0 & & & \\ \hline & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right)$$

In a general frame, this can be represented as

$$\sum_{i=1\cdot 3} \varepsilon_i^{\mu} \varepsilon_i^{\nu *} = -\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{m_W^2}\right)$$

The it is clear that we have missed a term in our expression for the squared amplitude. The missing term is

$$\frac{g^2}{2} \frac{1}{2} \operatorname{tr}\left[\not{k} \gamma^\mu \not{p} \gamma^\nu \cdot \frac{1}{2}\right] \cdot \frac{q^\mu q^\nu}{m_W^2}$$

which works out to

$$\frac{g^2}{8 m_W^2} \operatorname{tr}[\not{k} \not{q} \not{p} \not{q}] = \frac{g^2}{8 m_W^2} \cdot 4 \cdot [\, 2\, k\cdot q\; p\cdot q - k\cdot p\; q^2 \,]$$

$$= \frac{g^2}{2 m_W^2}\left[ m_t E_W (m_t^2 - m_W^2) - m_t E_p m_W^2 \right]$$

$$= \frac{g^2}{2\cdot 2 m_W^2} (m_t^2 - m_W^2) \left[ (m_t^2 + \cancel{m_W^2}) - \cancel{m_W^2} \right]$$

Note that this is larger than the term that we originally kept; it contains the parametric factor

$$m_t^2 / m_W^2 \sim 4$$

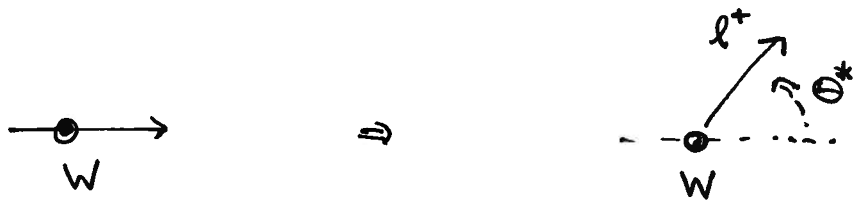
The final result for the width of the top quark is

$$\Gamma_t = \frac{\alpha_w}{16} m_t \frac{m_t^2}{m_W^2} \left(1 + 2\frac{m_W^2}{m_t^2}\right)\left(1 - \frac{m_W^2}{m_t^2}\right)^2$$

It probably would not surprise you to learn that the extra term is the contribution from longitudinally polarized $W$ bosons. In fact, you can show that

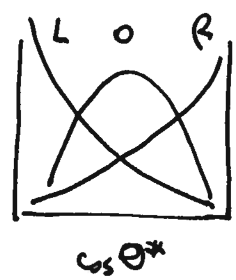
$$\frac{\Gamma(t \to W_0^+ b)}{\Gamma(t \to W^+ b)} = \frac{m_t^2/m_W^2}{m_t^2/m_W^2 + 2} = 70\%$$

This formula can be checked in the laboratory by observing fully reconstructed $t\bar{t}$ events. We select semileptonic events in which one $W$ decays to $\ell\nu$. Boost along the $W$ direction of motion to its rest frame. Then let $\theta^*$, the $W$ *helicity angle*, be the angle between the $W$ direction and the direction of the lepton in rest frame.




The helicity angle has the distribution

$$\begin{array}{ll} W_R^+ & (1+\cos\theta^*)^2 \\ W_0^+ & 2\sin^2\theta^* \\ W_L^+ & (1-\cos\theta^*)^2 \end{array}$$




for the various $W$ helicities. Then the helicity angle distribution in top decay is predicted to have the form


$$(\Gamma(t \to W^+_R b) = 0)$$

This distribution has been measured at the Tevatron and at the LHC, in good agreement with the Standard Model prediction. The figure shows measurements from CDF [4] and ATLAS [5]. The shapes are altered from the idea shape shown above due to detector effects. In particular, leptons at $\cos\theta^* = -1$ are slow in the lab frame and thus detected rather inefficiently. However, the data clearly shows that longitudinal $W$ polarizations dominate.

Why does the extra term from the correct expression for the $W$ polarization sum give so large a result? It is because the longitudinal $W$ polarization vector is peculiar. Let's analyze this for the case of a $W$ moving in the $\hat{3}$ direction. In the $W$ rest frame, the three polarization vectors are

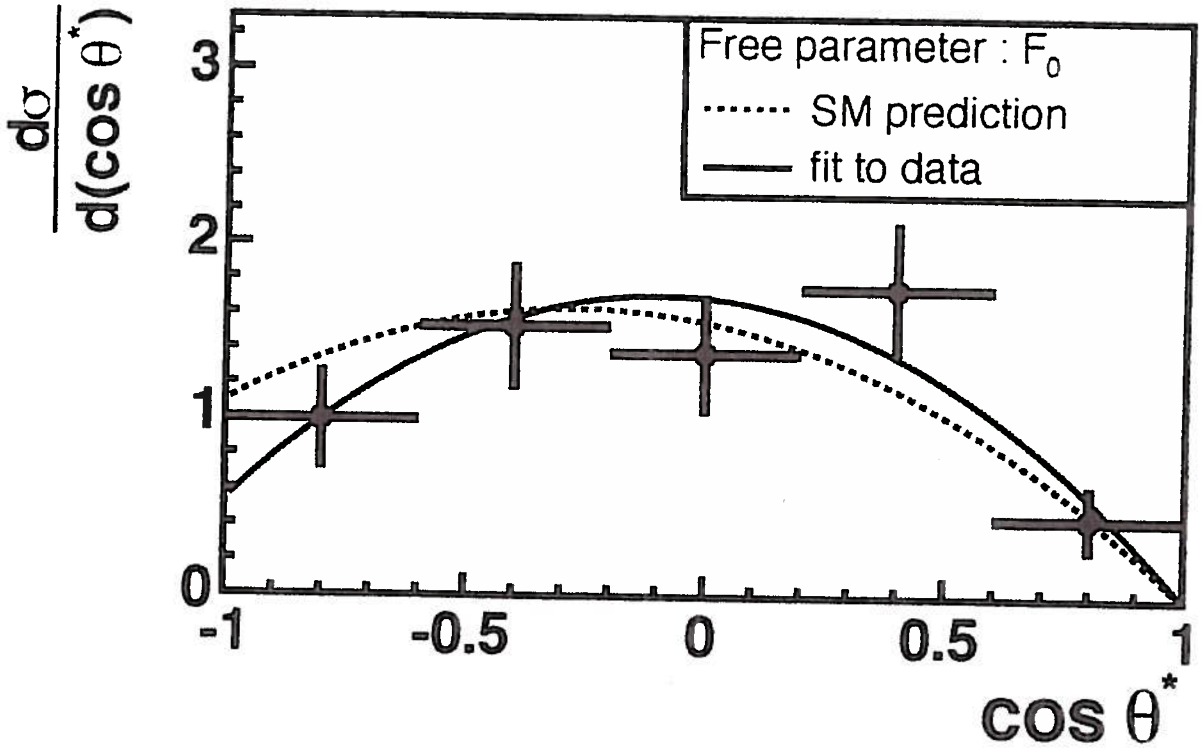
$$\epsilon_R^\mu = \frac{1}{\sqrt{2}}(0\ 1\ i\ 0) \qquad \epsilon_L^\mu = \frac{1}{\sqrt{2}}(0\ 1\ {-i}\ 0) \qquad \epsilon_0^\mu = (0\ 0\ 0\ 1)$$

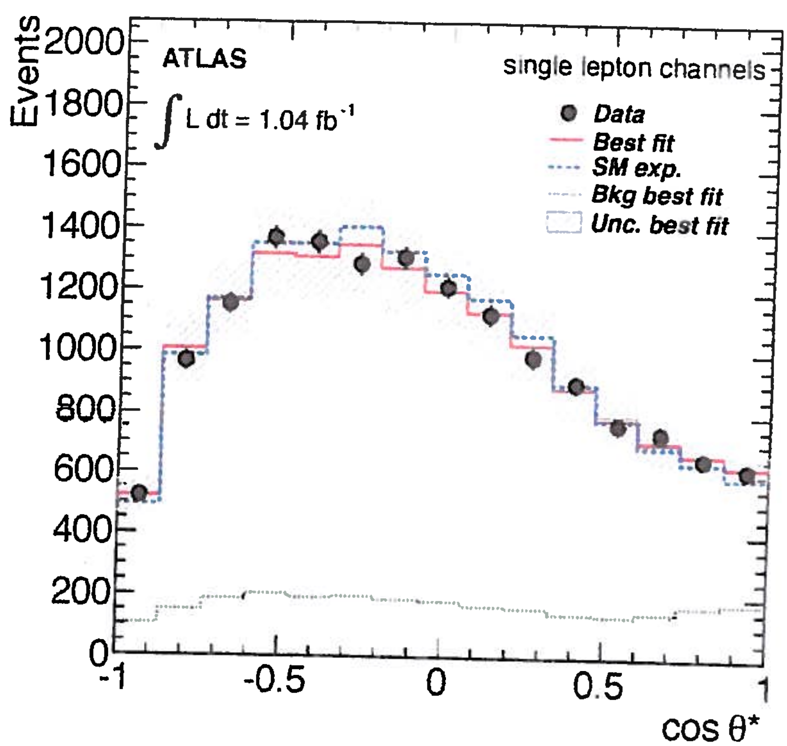
Now boost in the $\hat{3}$ direction. The vectors $\epsilon_R$, $\epsilon_L$ are unchanged. However, for $q^\mu = (E_q, 0, 0, q)$, the longitudinal polarization vector is

$$\epsilon_0^\mu = \left(\frac{q}{m_W}, 0, 0, \frac{E_q}{m_W}\right)$$

The components of the vector become arbitrarily large when $E_q \gg m_W$. This is an easy consequence of the Lorentz boost, but nevertheless it is decidedly weird.

It is instructive to check that this gives the correct result for top quark decay. I will now compute the amplitude for top quark emission of a longitudinally polarized $W$ boson. For simplicity (since we already have the full answer) I will work in the limit $m_t/m_W \gg 1$. The matrix element is




$\cos\theta^*$ distributions in $t \to W^+ b$ from CDF [ ] (2006) and ATLAS [ ] (2012)

$$\mathcal{M} = i\frac{g}{\sqrt{2}}\,\bar{u}_b(k)\,\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)u_t(p)\,\epsilon^*_{\mu 0}$$

Set

$$\epsilon^\mu_0 \approx \frac{q^\mu}{m_W} = \frac{(p-k)^\mu}{m_W}$$

in accord with the expression above. Then the matrix element evaluates to

$$\mathcal{M} = i\frac{g}{\sqrt{2}}\,\bar{u}(k)\,\frac{\not{p}-\not{k}}{m_W}\left(\frac{1-\gamma^5}{2}\right)u(p)$$

$$= i\frac{g}{\sqrt{2}}\,\bar{u}(k)\left(\frac{1+\gamma^5}{2}\right)\frac{\not{p}}{m_W}u(p) = i\frac{g}{\sqrt{2}}\,\bar{u}(k)\left(\frac{1+\gamma^5}{2}\right)u(p)\cdot\frac{m_t}{m_W}$$

and we see that it produces an extra factor of $m_t/m_W$. It is interesting that this factor

$$\frac{g}{\sqrt{2}}\frac{m_t}{m_W} = \sqrt{2}\,\frac{m_t}{v} \qquad \text{since} \qquad m_W = \frac{g}{2}v$$

is equal to the top quark Yukawa coupling

$$= y_t \qquad \text{since} \qquad m_t = \frac{y_t}{\sqrt{2}}v$$

We can now check the prediction of the GBET by comparing this result to the result we would have found for Goldstone boson emission. The Lagrangian term for that process is

$$\Delta\mathcal{L} = y_t\, \bar{Q}_{La}\, \varepsilon_{ab}\, \phi^*_b\, t_R = y_t\, \bar{b}_L\, \pi^- \left(\frac{1+\gamma^5}{2}\right) t + \cdots$$

which yields for the matrix element

$$\mathcal{M} = i\, y_t\, \bar{u}(k) \left(\frac{1+\gamma^5}{2}\right) u(p)$$

We find the same expression as above.

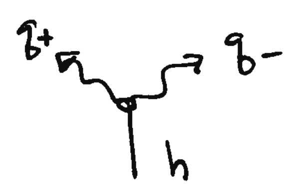
This exercise illustrates the GBET, and it also suggests another way to look at this theorem. Consider the limit in which the weak interaction gauge couplings $g$ and $g'$ are taken to zero, while the Yukawa couplings are kept at their physical values. Bjorken calls this the "gaugeless limit" of the Standard Model [6]. In this limit, the top quark will still decay. The dominant decay process will be emission of a Higgs sector particle

$$t \to \pi^+ b$$

In fact, this is just the Goldstone boson emission process considered in the previous paragraph.

The simplest application of the GBET to Higgs physics relates not to a Higgs with the physical mass but rather to a Standard Model Higgs boson with very large mass (a possibility allowed by the data until 2011). Consider the decay of such a particle to vector bosons

$$h \to W^+W^-,\ Z^0Z^0$$




The naive expectation for the Higgs boson width would be

$$\Gamma_h \sim \alpha_W m_h$$

The correct answer is quite different. The matrix element is

$$\mathcal{M} = i \frac{2m_W^2}{v} g^{\mu\nu} \varepsilon^{\mu *}(q_+) \varepsilon^{\nu *}(q_-) = i g m_W \varepsilon^*(q_+)\cdot\varepsilon^*(q_-)$$

Concentrate on the contribution of longitudinally polarized gauge bosons, using the approximation

$$\varepsilon(q) \approx \frac{q}{m_W}$$

Then the matrix element becomes

$$\mathcal{M} \cong i g m_W \frac{q_+ \cdot q_-}{m_W^2} \cong i \frac{g}{2} \frac{m_h^2}{m_W}$$

This leads to

$$\Gamma_h = \frac{1}{2m_h} \frac{1}{8\pi} \frac{g^2}{4} \frac{m_h^4}{m_W^2}$$

or

$$\Gamma_h = \frac{\alpha_W}{16} m_h \frac{m_h^2}{m_W^2}$$

Since

$$\frac{g}{2}\frac{m_h^2}{m_W} = \frac{g}{2}\frac{2\lambda v^2}{\frac{g}{2}v} = 2\lambda v$$

this expression is the same as the expression for the decay of a Higgs boson to two Goldstone bosons through the interaction term

$$\Delta\mathcal{L} = -2\lambda v h\, \pi^+ \pi^- \quad \Leftarrow \quad \Delta\mathcal{L} = -\lambda(|\phi|^2)^2$$

The complete expression for the decay width is

$$\Gamma(h \to W^+W^-) = \frac{\alpha_w}{16}\frac{m_h^3}{m_W{}^2}\left(1 - \frac{4m_W^2}{m_h^2}\right)^{\frac{1}{2}}\left(1 - \frac{4m_W^2}{m_h^2} + 12\frac{m_W^4}{m_h^4}\right)$$

For $m_h \gg m_W$, the width is enhanced over the naive expectation by a factor

$$\frac{m_h^2}{m_W^2}$$

Then, for example, a Standard Model Higgs boson of mass 1 TeV would also have a width of 1 TeV.

Even for decay to off-shell vector bosons at $m_h = 125$ GeV, the enhancement of the decay to longitudinally polarized vector bosons is nontrivial. This allows us to distinguish the coupling predicted in the Standard Model

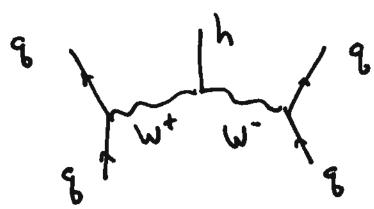
$$\Delta\mathcal{L} = g m_W\, h\, W_+^\mu W_{-\mu}$$

from couplings predicted in models where the Higgs boson is not associated with spontaneous symmetry breaking, for example, models where the 125 GeV resonance is purely a dilaton,

$$\Delta\mathcal{L} = A\, h\, F_{\mu\nu}(W^+) F^{\mu\nu}(W^-)$$

The latter coupling produces transversely polarized vector bosons only. At the moment, the second hypothesis is disfavored by the LHC data, but only by 1.7 $\sigma$ [7]. I expect that the gap will widen with more data from the next LHC run.

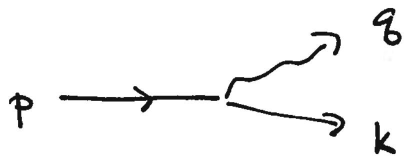
The $WW$ coupling to the Higgs boson is the central ingredient in the process of Higgs boson production by vector boson fusion.




This process raises an interesting question. The analysis above implies that, in $WW$ fusion, the Higgs boson should be produced dominantly from the state in which both $W$ bosons are longitudinally polarized. However, the GBET seems to raise a question about this. It seems to imply that high-momentum longitudinally polarized gauge bosons cannot be radiated by light quarks, since the coupling would be proportional to the (tiny) quark Yukawa couplings. Fortunately, this is not correct. If the $W$ boson is emitted from the quark in an almost collinear direction, the emission is not a high energy process but rather is characteristic of the $W$ mass scale. Then two light quarks can emit longitudinally polarized $W$ bosons, which can then interact with one another a Higgs sector Goldstone bosons.

It is instructive to work out the formulae for $W$ boson emission from quarks in the collinear limit. The analysis was first done by Dawson, in a very illuminating paper [8].

We wish to consider the collinear emission of a $W$ boson by a quark



To a first approximation, the momenta for the three particles can be written

$$p = (E, 0, 0, E)$$

$$k = ((1-z)E, -q_T, 0, (1-z)E) \qquad 0<z<1$$

$$q = (zE, q_T, 0, zE)$$

where $q_T$ is the transverse momentum of the $W$ relative to the quark. If this process is part of the $WW$ fusion process above, the initial and final quarks will be on shell, while the $W$ boson will be off-shell. Then, to order $q_T^2$, the momenta are

$$p = (E, 0, 0, E)$$

$$k = \left((1-z)E, -q_T, 0, (1-z)E - \frac{q_T^2}{2(1-z)E}\right)$$

$$q = \left(zE, q_T, 0, zE + \frac{q_T^2}{2(1-z)E}\right)$$

The denominator of the off-shell $W$ propagator is

$$q^2 - m_W^2 = -q_T^2 - \frac{z q_T^2}{(1-z)} - m_W^2 = -\left(\frac{q_T^2}{(1-z)} + m_W^2\right)$$

The matrix element for the emission process is

$$\mathcal{M} = \frac{g}{\sqrt{2}}\, \bar{u}(k)\, \bar{\sigma}\cdot\varepsilon_W^*\, u(p)$$

In the limit of zero mass quarks, only left-handed polarized initial and final quarks participate. These have polarization spinors

$$u(p) = \sqrt{2E}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad u(k) = \sqrt{2(1-z)E}\begin{pmatrix} q_T/2(1-z)E \\ 1 \end{pmatrix}$$

The polarization vectors for the emitted $W$ boson are

$$\varepsilon^\mu_{R,L} = \frac{1}{\sqrt{2}}\left(0\,, 1\,, \pm i\,, -\frac{q_T}{zE}\right)$$

$$\varepsilon^\mu_0 = \left(\frac{q}{m_W}\,, \frac{q_T}{m_W}\,, 0\,, \frac{zE}{m_W}\right)$$

Plugging these polarization vectors into the matrix element, we find, for right-handed $W$ polarization,

$$\mathcal{M} = \frac{g}{\sqrt{2}}\,\frac{2E}{\sqrt{2}}\sqrt{1-z}\cdot\frac{q_T}{zE}$$

for left-handed $W$ polarization,

$$\mathcal{M} = \frac{g}{\sqrt{2}}\,\frac{2E}{\sqrt{2}}\sqrt{1-z}\left(\frac{q_T}{zE} + \frac{q_T}{(1-z)E}\right)$$

and for longitudinal $W$ polarization, $\quad q = \left((zE)^2 - m_W^2\right)^{1/2}$

$$\mathcal{M} = \frac{g}{\sqrt{2}}\,2E\sqrt{1-z}\cdot\frac{q-zE}{m_W} = \frac{g}{\sqrt{2}}\,2E\sqrt{1-z}\cdot\left(\frac{-m_W^2/zE}{m_W}\right)$$

In all, the matrix elements are

$$\mathcal{M} = \begin{cases} g\sqrt{1-z}\,\dfrac{q_T}{z} & R \\[2ex] g\sqrt{1-z}\,\dfrac{q_T}{z(1-z)} & L \\[2ex] g\dfrac{\sqrt{1-z}}{\sqrt{2}}\,\dfrac{m_W}{z} & 0 \end{cases}$$

Consider a process with $W$ emission from a quark on one side. The full reaction is $uX \to dY$. We would like to relate the cross section for this reaction to that for the reaction $WX \to Y$,


The full cross section is

$$\sigma(uX \to dY) = \frac{1}{2s}\int \frac{d^3k}{(2\pi)^3\, 2k}\int d\Pi_Y \left|\frac{\mathcal{M}(u\to dW^+)\ \mathcal{M}(W^+X\to Y)}{q^2-m_W^2}\right|^2$$

Replacing the integral over $k$ by an integral over the variables $z$ and $q_T$, we find

$$\sigma(uX\to dY) = \frac{1}{2s}\int \frac{dz\, p\ \pi\, dq_T^2}{8\pi^3\, 2(1-z)\,p}\int d\Pi_Y \frac{|\mathcal{M}(u\to dW^+)|^2\,|\mathcal{M}(W^+X\to Y)|^2}{\left[m_W^2 + q_T^2/(1-z)\right]^2}$$

$$= \int dz\, \frac{1}{(4\pi)^2}\int dq_T^2\, \frac{1}{\left[q_T^2+(1-z)m_W^2\right]^2}\, z(1-z)\,|\mathcal{M}(u\to dW)|^2 \cdot \frac{1}{2\hat{s}}\int d\Pi_Y\, |\mathcal{M}(W^+X\to Y)|^2$$

Finally, this becomes

$$(\hat{s} = zs)$$

$$= \frac{1}{(4\pi)^2}\int dz \int \frac{d(q_T^2)}{\left[q_T^2+(1-z)m_W^2\right]^2}\, z(1-z)\,|\mathcal{M}(u\to dW)|^2 \cdot \sigma(W^+X\to Y)$$

The $W$ emission factor has the form of a parton distribution

$$\sigma(uX \to dY) = \int_0^1 dz\; f_W(z)\, \sigma(W^+(zp)X \to Y)$$

with, for the three polarization states

$$f_W(z) = \begin{cases} \dfrac{\alpha_w}{4\pi} \displaystyle\int \frac{dq_T^2\, q_T^2}{[q_T^2 + (1-z) m_W^2]^2} \, \frac{(1-z)^2}{z} & R \\[2ex] \dfrac{\alpha_w}{4\pi} \displaystyle\int \frac{dq_T^2\, q_T^2}{[q_T^2 + (1-z) m_W^2]^2} \, \frac{1}{z} & L \\[2ex] \dfrac{\alpha_w}{8\pi} \displaystyle\int \frac{dq_T^2\, m_W^2}{[q_T^2 + (1-z) m_W^2]^2} \, \frac{(1-z)^2}{z} & 0 \end{cases}$$

For the transverse polarizations, the integral over $q_T$ is logarithmic up to $q_T \sim s$, and we find

$$f_{W,R+L}(z) \cong \frac{\alpha_w}{4\pi} \cdot \log \frac{s}{m_W^2} \cdot \frac{1 + (1-z)^2}{z}$$

which has the familiar form of a Weizsacker-Williams distribution. For longitudinally polarized $W$ bosons, this integral is finite and gives

$$f_{W,0}(z) \cong \frac{\alpha_w}{8\pi} \, \frac{(1-z)}{z}$$

Nevertheless, the result is substantial.

This results generalizes to give an important method for studying the Higgs sector at high-energy colliders. Light quarks (or leptons) at high energy can radiate longitudinally polarized $W$ bosons, and these bosons then interact like Goldstone bosons of the Higgs sector. If there are new strong interactions in the Higgs sector, these must show themselves in high energy $WW$ scattering.

The reaction $gg \to ZZ$ has been in the news recently in relation to a proposal by Caola and Melnikov that this process can be used to measure the Higgs boson width

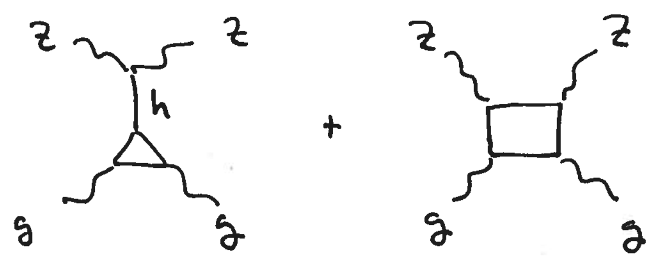
[9]. The data analysis has been carried out by the CMS Collaboration, which claims an upper bound on the Higgs width of 22 MeV, only 5.4 times the expected value in the Standard Model [10]. This quite a striking result, since a measurement of the very small value expected for the Higgs width had been considered inaccessible to collider experiments. One might, however, raise the question, is this determination true in all models or dependent upon some hidden assumption. In a recent paper, Englert and Spannowsky have clarified this point [11]. The story gives an interesting application of the GBET.

Here is the original argument of Caola and Melnikov. Higgs boson production at the LHC is dominated by the process $gg \to h$. When we measure the rate of Higgs production at the LHC with decay to $ZZ^*$, we are measuring a quantity proportional to

$$\frac{\Gamma(h \to gg)\,\Gamma(h \to ZZ)}{\Gamma_h}$$

where $\Gamma_h$ is the total width of the Higgs boson. The LHC results agree with the Standard Model prediction to about 20%. These results could indicate that all three factors are in good agreement with the Standard model predictions, but it is possible that both the numerator and the denominator are different from their Standard Model values.

Caola and Melnikov proposed to check this by studying the reaction $gg \to ZZ$ at very high values of the $ZZ$ invariant mass ($m(ZZ) > 400$ GeV). There are two important classes of Feynman diagrams,




The first includes the Higgs boson coupling to 2 gluons; the second is a set of box diagrams that do not involve the Higgs boson. The box diagrams—and the background process $q\bar{q} \to ZZ$—are dominated by production of transversely polarized gluons. However, by selecting $Z$ decays with $\cos\theta^*$ near 0 on both sides, we can select for events with longitudinal $Z$ polarization. In the CMS analysis, this is done using a classifier called MELA, described in [7]. From here on, I will assume, for simplicity, that it is possible to measure the cross section for purely longitudinal $Z$

pair production, $gg \to Z_0 Z_0$. In this case, in the Standard Model, the box diagrams are dominated by top quark loops.

Caola and Melnikov assume that one can vary the Higgs width or the value of the numerator in the previous equation while the box diagrams remain fixed at their Standard Model value. They then note, that, at high energy, the diagrams of the first type above are proportional to

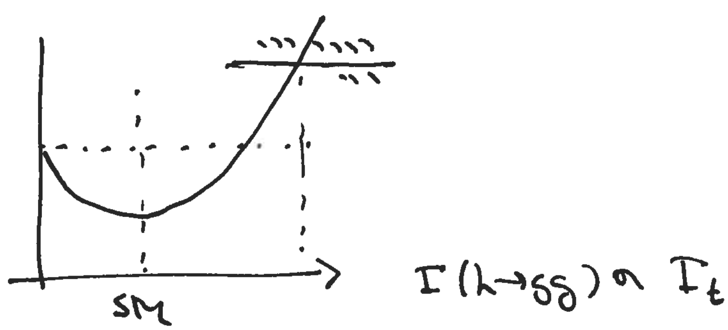
$$\frac{\Gamma(h \to gg)\, \Gamma(h \to ZZ)}{s}$$

with the squared center of mass energy $s$ replacing $\Gamma_h$. Then comparison of the measurements for an on-shell Higgs boson with the cross section for an off-shell Higgs boson gives the Higgs width.

The power of the argument comes from the fact that the Higgs diagrams contributing to $gg \to Z_0 Z_0$ contain a factor that increases rapidly as $s$ becomes large. In fact, as we saw earlier, the $h \to ZZ$ amplitude behaves as


$$\sim \frac{2m_Z^2}{v}\,\frac{q_1 \cdot q_2}{m_Z^2} \sim \frac{m_Z^2}{v} \cdot \left(\frac{s}{m_Z^2}\right)$$

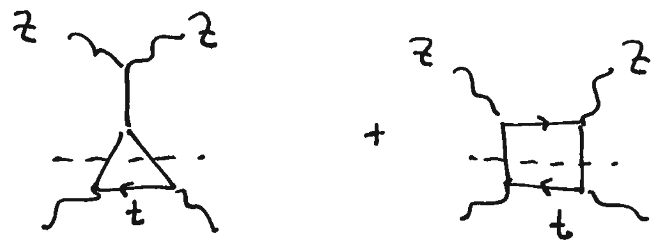
for $s >> m_Z^2$. In turns out that there is destructive interference between the Higgs diagrams and the box diagrams. Then, as a function of $\Gamma_t$ or $\Gamma(h \to gg)$, the cross section $\sigma(gg \to Z_0 Z_0)$ behaves as



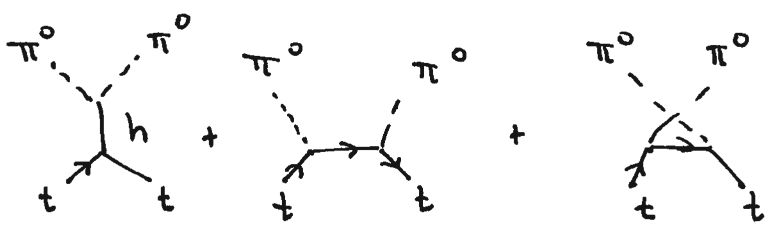

. An upper bound on the cross section puts a limit on $\Gamma_t$.

There is something odd about this argument, though. The GBET states that the cross section for $gg \to Z_0 Z_0$ should become equal to the cross section for $gg \to \pi^0\pi^0$ at high energies, where $\pi^0$ is the Goldstone boson in the Higgs multiplet, and this latter cross section does not contain a factor $s/m_Z^2$ but rather $m_h^2/m_Z^2 \sim \lambda/g^2$. Something is missing.

The paradox was explained, actually, in the 1980's, when Glover and van der Bij first computed the cross section for $gg \to ZZ$ [12]. They found the destructive interference between the two classes of diagrams and interpreted it as a consequence of the GBET. Consider, in particular, the imaginary part of the amplitude, which arise from the unitarity cuts




The sum of diagrams on top of the cut is gives the amplitude for $t\bar{t} \to ZZ$. A well-known property of this amplitude is that the individual diagrams behave as $s/m_Z^2$ but that the sum of diagrams is smaller and is equal to the amplitude arising from




Such a cancellation is called a *unitarity cancellation*, since the individual diagrams grow to fast with $s$ to respect the upper bound from unitary, while the sum, after the cancellation, respects this bound. A similar cancellation in the process $e^+e^- \to W^+W^-$ is described in Section 21.3 of the Peskin and Schroeder textbook.

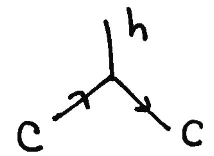
Indeed, if we enhance $\Gamma(h \to gg)$ by adding contributions from new heavy quarks, those quarks will also contribute to the box diagrams. The two sets of new contributions will have a cancellation just like the one for the top quark loops. The whole curve for $\sigma(gg \to Z_0Z_0)$ drawn above will be shallower and will yield a weaker limit on $\Gamma_t$.

However, as Englert and Spannowsky point out, there is a case in which the Caola-Melnikov limit on $\Gamma_t$ applies without change. Assume that we add to the Standard Model a set of colored scalar fields with no electromagnetic or weak charge. Assume that these particles obtain mass from a term

$$\Delta \mathcal{L} = -m_c^2 \, |\phi|^2 \, |C|^2$$

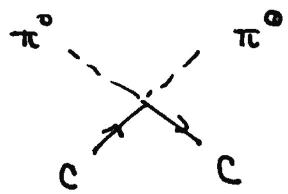
where $\phi$ is the Higgs field and $C$ is the colored scalar field. Then, obviously, only the Higgs diagrams are affected and the Caola-Melnikov argument goes through.

Can we understand this from the point of view of the GBET? The contribution of the $C$ scalars to the Higgs diagram involves the vertex



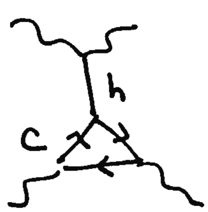

$$= 2 \frac{m_c^2}{v}$$

But, the Lagrangian I have written also has a vertex between two $C$ bosons and two Goldstone bosons,




$$= \frac{2 m_c^2}{v^2}$$

The full contribution of $C$ to the Higgs diagram has the form




$$= (\text{triangle diagram}) \cdot 2 \frac{m_c^2}{v} \cdot \frac{1}{s} \cdot \frac{2 m_Z^2}{v} \, \frac{s}{2 m_Z^2}$$

The Goldsone boson emission diagram has the form


$$= (\text{triangle diagram}) \cdot \frac{2 m_c^2}{v^2}$$

In fact, in the limit $s \gg m_Z^2$, these diagrams are equal, as required by the GBET. In this case, then, the scalar field Lagrangian produces a new term contributing the

Goldstone boson emission that explains the new term that appears in double $Z$ boson production.

I understand that there is much interest in China today in the study of physics processes at a 100 TeV $pp$ collider. Such a collider will have important things to say about the nature of the Higgs sector at high energies, hopefully giving us insight into the cause of electroweak symmetrybreaking. At such high energies, we can ignore the $W$ mass and treat the $W$ as a transversely polarized gauge boson—plus a Higgs sector Goldstone boson. The ideas discussed in this lecture will be very important in analyzing measurements of high energy $W$ and $Z$ scattering, and in using these processes to shed light on the most important open problems of particle physics

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